ELEG5481

SIGNAL PROCESSING OPTIMIZATION TECHNIQUES

8. SEMIDEFINITE PROGRAM

Semidefinite Programming (SDP)

Inequality form:

$$\begin{aligned} &\min\ c^T x \\ &\ \text{s.t.}\ F(x) \preceq 0 \end{aligned}$$

where $F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n$, $F_i \in \mathbf{S}^{p\times p}$.

Standard form:

$$\begin{aligned} &\min\ \mathbf{tr}(CX) \\ &\ \text{s.t.}\ X \succeq 0 \\ &\qquad \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \end{aligned}$$

where $A_i \in \mathbf{S}^{n\times n}$, and $C \in \mathbf{S}^{n\times n}$.

- The inequality & standard forms can be shown to be equiv.
- $F(x) \preceq 0$ is commonly known as a **linear matrix inequality** (LMI).
- An SDP with multiple LMIs

$$\begin{aligned} &\min\; c^T x \\ &\text{ s.t. } F_i(x) \preceq 0, \quad i = 1, \ldots, m \end{aligned}$$

can be reduced to an SDP with one LMI since

$$F_i(x) \preceq 0,\; i = 1, \ldots, m \Leftrightarrow \mathbf{blkdiag}(F_1(x), \ldots, F_m(x)) \preceq 0$$

where **blkdiag** is the block diagonal operator.

Example: Max. eigenvalue minimization

Let $\lambda_{\max}(X)$ denote the maximum eigenvalue of a matrix X.

Max. eigenvalue minimization problem:

$$\min_{x} \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n$.

We note that fixing x,

$$\lambda_{\max}(A(x)) \leq t \Longleftrightarrow A(x) - tI \preceq 0$$

Hence, the problem is equiv. to

$$\begin{aligned} &\min_{x,t} \; t \\ &\text{s.t. } A(x) - tI \preceq 0 \end{aligned}$$

LP as SDP

Standard LP:

$$
\begin{aligned}
\min\ & c^T x \\
\text{s.t. } & x \succeq 0, \\
& a_i^T x = b_i, \quad i = 1, \ldots, m
\end{aligned}
$$

Let $C = \mathbf{diag}(c)$, & $A_i = \mathbf{diag}(a_i)$. The standard SDP

$$
\begin{aligned}
\min\ & \mathbf{tr}(CX) \\
\text{s.t. } & X \succeq 0 \\
& \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m
\end{aligned}
$$

is equiv. to the LP since $X \succeq 0 \Longrightarrow \mathbf{diag}(X) \succeq 0$.

Inequality form LP

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } Ax \preceq b \end{aligned}$$

is equiv. to the SDP

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } \mathbf{diag}(Ax - b) \preceq 0 \end{aligned}$$

because $X \succeq 0 \Longrightarrow X_{ii} \geq 0$ for all i.

Schur Complements

Let $X \in \mathbf{S}^n$ and partition

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$S = C - B^T A^{-1} B$ is called the **Schur complement** of A in X (provided $A \succ 0$).

Important facts:

- $X \succ 0$ iff $A \succ 0$ and $S \succ 0$.
- If $A \succ 0$, then $X \succeq 0$ iff $S \succeq 0$.

Schur complements are useful in turning some nonlinear constraints into LMIs:

Example: The convex quadratic inequality

$$(Ax + b)^T (Ax + b) - c^T x - d \leq 0$$

is equivalent to

$$\begin{bmatrix} I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succeq 0$$

QCQP as SDP

A convex QCQP can always be written as

$$\begin{aligned} \min \ & \|A_0 x + b_0\|_2^2 - c_0^T x - d_0 \\ \text{s.t.} \ & \|A_i x + b_i\|_2^2 - c_i^T x - d_i \leq 0, \quad i = 1, \ldots, L \end{aligned}$$

By Schur complement, the QCQP is equiv. to

$$\begin{aligned} \min \ & t \\ \text{s.t.} \ & \begin{bmatrix} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0^T x + d_0 + t \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, L \end{aligned}$$

Example: The second order cone inequality:

$$\|Ax + b\|_2 \leq f^T x + d$$

If the domain is such that $f^T x + d > 0$, the inequality can be re-expressed as

$$f^T x + d - \frac{1}{f^T x + d}(Ax + b)^T (Ax + b) \geq 0.$$

By Schur complement, the inequality is equiv. to

$$\begin{bmatrix} (f^T x + d)I & Ax + b \\ (Ax + b)^T & f^T x + d \end{bmatrix} \succeq 0$$

- This result indicates that SOCP can be turned to an SDP.

Example: 2-norm minimization

- Consider

$$\min_x \|A(x)\|_2$$

where $A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n$, with $A_1, \ldots, A_n \in \mathbf{R}^{p \times q}$.

- This problem can be reformulated as an SDP. Here is the trick:

$$\begin{aligned} \|A(x)\|_2 \le t &\iff \|A(x)\|_2^2 \le t^2, \; t \ge 0 \\ &\iff A^T(x)A(x) \preceq t^2 I, \; t \ge 0 \\ &\iff \begin{bmatrix} tI & A(x) \\ A^T(x) & tI \end{bmatrix} \succeq 0 \end{aligned}$$

- Hence the min. 2-norm problem can be written as

$$\begin{aligned} &\min_{x,t} \; t \\ &\text{s.t.} \begin{bmatrix} tI & A(x) \\ A^T(x) & tI \end{bmatrix} \succeq 0 \end{aligned}$$

Applications of SDP

- There are many applications for SDP.
- SDP has been used to
  - do robust control;
  - deal with finite representation of semi-infinite constraints (like those in filter designs);
  - deal with robust problems such as robust SOCP, robust SDP, ...
  - approximate a host of nonconvex quadratic optimization problems;
  - handle certain advanced transceiver designs arising in MIMO communications;
  - handle training problems in pattern classification;
  - and many more...

Some References of SDP Applications (probably a small fraction)

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